HAD20+2

The finite Fourier Transform and projective 2-designs

Topics in Quantum Design Theory

Gerhard Zauner, Kraków, 29.06.2022

Abstract

A basis of certain **eigenvectors** of the **finite Fourier Transform** in some dimensions is used to generate **„Weyl-Heisenberg covariant projective** 2**-designs“***.

**) like SICs and complete sets of MUBs.*

Outline

- Motivation: The Quantum Harmonic Oscillator
- Weyl-Heisenberg Matrices and Fourier Matrix
- Projective 2-designs
- An example for $d = 3$ and some numerical search
- The Clifford Group
- The Main Result
- Sketch of the proof
- Another „Measurement in the Sky“
- Concluding remarks

Motivation: The Quantum Harmonic Oscillator

$(\mathbf{X}f)(x) = xf(x)$

Position operator

$(\mathbf{P}f)(x) = -i\frac{d}{dx}f(x)$

Momentum operator

$$\mathbf{H} = \frac{1}{2}(\mathbf{X}^2 + \mathbf{P}^2)$$

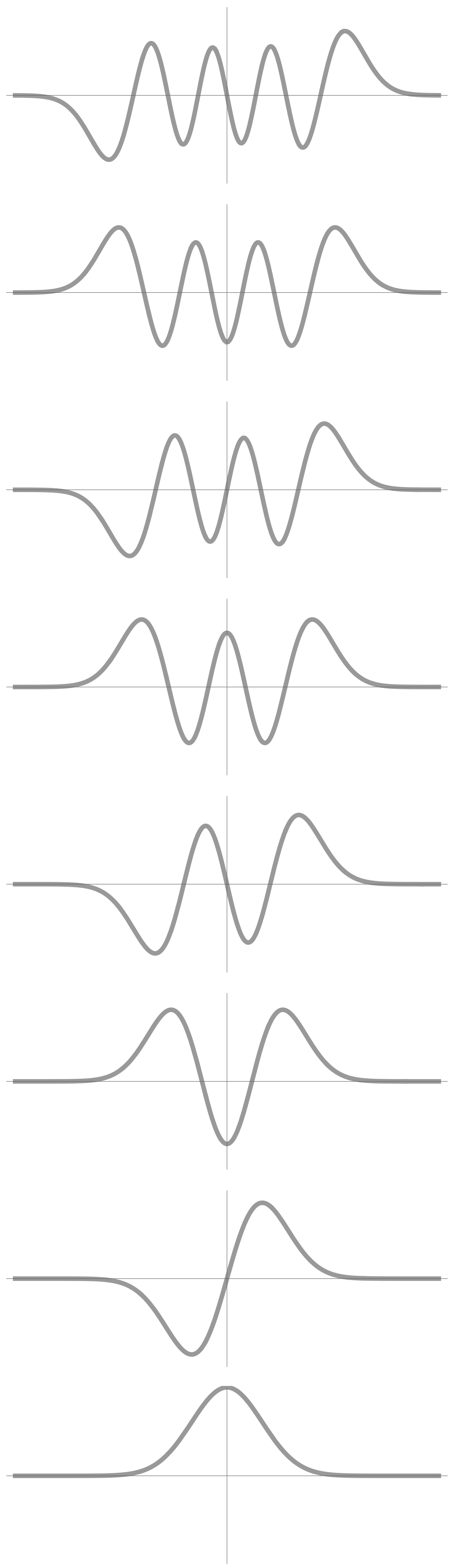
$$(\mathbf{F}f)(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i2\pi xy}f(x)dx$$

Fourier-Transform

Eigenstates

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}H_n(x)e^{\frac{-x^2}{2}}$$

$H_n(x)$ Hermitian polynomials



Are there analog structures in finite dimensional Hilbertspaces?

The Weyl-Heisenberg Matrices

$\mathbb{C}^d$

Infinite case: $\mathbf{X}, \mathbf{P} \leftrightarrow \mathbf{W}(r,s) = e^{i(r\mathbf{P}+s\mathbf{X})} = e^{-\frac{irs}{2}} e^{ir\mathbf{P}} e^{is\mathbf{X}}$

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & e^{\frac{i2\pi}{d}} & 0 & \dots & 0 \\ 0 & 0 & e^{\frac{i4\pi}{d}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{\frac{i2(d-1)\pi}{d}} \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

$$\left.\begin{aligned} \tau &= e^{\frac{i\pi(d+1)}{d}} = -e^{\frac{i\pi}{d}} \\ r, s &\in \mathbb{Z}_d / \mathbb{Z}_{2d} \end{aligned}\right\}$$

$$\mathbf{W}_{(r,s)} = \tau^{rs} \mathbf{V}^r \mathbf{U}^s$$

But no (unique) infinitesimal generators.

Remarks to the notation:

Generalized Pauli Matrices $\left\{ \begin{aligned} \mathbf{X} &\leftrightarrow \mathbf{V} \\ \mathbf{Z} &\leftrightarrow \mathbf{U} \end{aligned} \right\}$ Notation due to Schwinger

Displacement $\mathbf{D}_{(r,s)} \leftrightarrow \mathbf{W}_{(r,s)}$ for Weyl

The Fourier Matrix

$$\mathbf{F} = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{\frac{2i\pi}{d}} & e^{\frac{4i\pi}{d}} & \dots & e^{\frac{2(d-1)i\pi}{d}} \\ 1 & e^{\frac{4i\pi}{d}} & e^{\frac{8i\pi}{d}} & \dots & e^{\frac{4(d-1)i\pi}{d}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{\frac{2(d-1)i\pi}{d}} & e^{\frac{4(d-1)i\pi}{d}} & \dots & e^{\frac{(d-1)^2 i\pi}{d}} \end{pmatrix}.$$

The multiplicity of the eigenvalues as function of dimension d :

	1	-1	i	-i
d=4k	k+1	k	k	k-1
d=4k+1	k+1	k	k	k
d=4k+2	k+1	k+1	k	k
d=4k+3	k+1	k+1	k+1	k

*There is a long and ongoing history to define an **eigenvector decompositions** of the Fourier matrix, which is in some sense unique, and resembles the eigenstates of the quantum harmonic oscillator.*

Tiny selection of papers on eigenvectors of the Fourier Matrix

by Mathematicians, Physicists, and Electrical and Electronics Engineers (from last 50 years)

- McClellan, J.H. and Parks, T.W.: Eigenvalue and Eigenvector Decomposition of the Discrete Fourier Transform. (1972)
- Auslander, L. and Tolimieri, R.: Is computing with the finite Fourier Transform pure or applied mathematics? (1979)
- Morton, P.: On the Eigenvectors of Schur's Matrix. (1980),
- Dickinson, B.W., and Steiglitz, K.: Eigenvectors and functions of the discrete Fourier transform. (1982)
- Grünbaum, F.A.: The eigenvectors of the discrete Fourier transform: A version of the Hermite functions. (1982)
- Balian, R., and Itzykson, C.: Observations sur la mécanique quantique finie. (1986)
- Zhang, S and Vourdas, A.: Analytic Representation of Finite Quantum Systems. (2005)
- Gurevich, S., and Hadani, R.: On the diagonalization of the discrete Fourier transform. (2008)
- Kuznetsov A., and Kwasnicki M.: Minimum Hermite-Type eigenbasis of the discrete Fourier transform. (2017)

Now

*) See also papers about „*MUB balanced states*“ and „*Minimum uncertainty states*“

The approaches are either starting with the continuous harmonic oscillator (e.g. sampling at equidistant points), or focusing on algebraic methods (using e.g. the Clifford group).

Here we propose a characterization on terms of Quantum Designs, specifically ***projective 2-designs*** *for some dimensions.*

(Projective) 2-designs

Let $\{\,|\psi_i\rangle : 1 \le i \le N\}$ be N normed vectors and $\{\mathbf{P}_i := |\psi_i\rangle\langle\psi_i| : 1 \le i \le N\}$ be the corresponding projection matrices. There are many equivalent definitions, when these sets are a projective 2-design, including

$$\frac{1}{N}\sum_{i=1}^{N} \mathbf{P}_i \otimes \mathbf{P}_i = \frac{2}{d(d+1)}\mathbf{\Pi}_{sym}$$

Where $\mathbf{\Pi}_{sym}$ is the orthogonal projection on the symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$

or

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} (tr(\mathbf{P}_i\mathbf{P}_j))^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} |\langle\psi_i|\psi_j\rangle|^4 \ge \frac{2}{d(d+1)}$$

becomes ***equality***.

Well known examples:

- **SIC**s: (for all d ?) $N = d^2$
- **MUB**s: complete sets for d primepower $N = d(d+1)$
- Clifford group applied to any vector (for d prime) $N = d(d^2-1)$

An example, $d = 3$

$$\mathbf{F} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix}, \qquad \alpha = e^{2\pi i/3}.$$

has the 3 eigenvalues ± 1 and i, and the (up to phases) *unique* normed eigenvectors

$$|\psi_1\rangle = \frac{1}{\sqrt{6+2\sqrt{3}}} \begin{pmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{pmatrix}, \quad |\psi_{-1}\rangle = \frac{1}{\sqrt{6-2\sqrt{3}}} \begin{pmatrix} 1-\sqrt{3} \\ 1 \\ 1 \end{pmatrix}, \quad |\psi_i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

We apply the $d^2 = 9$ Weyl-Heisenberg Matrices to each of the $d = 3$ vectors and get the set of $d^3 = 27$ vectors $\{\mathbf{W}_{(r,s)}|\psi_x\rangle : x = \pm 1, i$ and $0 \leq r, s, \leq 2\}$.

These vectors form a 2-design.

Numerical search

Numerical search found no complete orthonormal eigenvector basis of the Fourier matrix for each $d = 4,5,6$, that generates a 2-design like above.

But for $d = 7,11$ there were found in each case (seemingly unique) solutions.

For $d = 5$ there is a unique base of 2 eigenvectors of eigenvalue 1, which together with the unique eigenvector for eigenvalue -1 generates a set of $3d^2 = 75$ vectors, that form a 2-design.

In the next slides we construct the solutions for prime numbers $d = 4k + 3$ and sketch the prove of the 2-design property.

The Clifford Group

The *Clifford Group* is defined as the *normalizer* of the Weyl-Heisenberg group in the group of unitary matrices $\mathbf{U}$. It is a projective representation of $SL(2,\mathbb{Z}_d)$ (we assume d to be odd).

$$\mathbf{U}\mathsf{W}_{(r,s)}\mathbf{U}^{-1} = \mathsf{W}_{(r',s')} \quad \Longleftrightarrow \quad \begin{bmatrix} r' \\ s' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,\mathbb{Z}_d), r, s, r', s' \in \mathbb{Z}_d$$

The Fourier Matrix $\mathbf{F}$ is an element of the Clifford group. The subgroup of elements of the Clifford Group, that **commute with** $\mathbf{F}$ are the projective representations of

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \in SL(2,\mathbb{Z}_d) \quad \Longleftrightarrow \quad \alpha^2 + \beta^2 = 1$$

This group is *abelian* and for *odd primes* d a *cyclic group* of order $\begin{cases} d-1 & \text{if} \quad d = 4k+1 \\ d+1 & \text{if} \quad d = 4k+3. \end{cases}$

Representations of the Clifford Group.

(we assume d to be an odd prime)

A projective (metaplectic) representation for the subgroup of elements commuting with $\mathbf{F}$, except for the identity ($\beta = 0 \Leftrightarrow \alpha = \pm 1$), in the standard basis $|\mathbf{e}_r\rangle$, with $\mathbf{r} \in \mathbb{Z}_d$ is e.g. given by Appleby as

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \longrightarrow \frac{e^{i\theta}}{\sqrt{d}} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \tau^{\left(\frac{1}{\beta}\right)(\alpha r^2 - 2rs + \alpha s^2)} |\mathbf{e}_r\rangle\langle\mathbf{e}_s|$$

The Weyl-Heisenberg matrices are an orthogonal basis of all matrices. Therefore we can also expand the elements above in this basis. An according representation, except for the identity, is e.g. given by Balian, Itzykson, Athanasiu, Floratos, Nicolis as

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \longrightarrow \frac{e^{i\eta}}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \tau^{\frac{\beta}{2(1-\alpha)}(r^2+s^2)} \mathsf{W}_{(r,s)}$$

The overall phases factors $e^{i\theta}$ rsp. $e^{i\eta}$ can be chosen such, that the presentation becomes de-projectivized (ordinary rsp. faithfull). We don't need this here, and set the factors to 1.

Eigenvectors of the Fourier matrix for prime $d = 4k + 3$

First we simplify the representation

$$\frac{1}{d}\sum_{r=0}^{d-1}\sum_{s=0}^{d-1}\tau^{\frac{\beta}{2(1-\alpha)}(r^2+s^2)}\mathsf{W}_{(r,s)} \qquad = \frac{1}{d}\sum_{r=0}^{d-1}\sum_{s=0}^{d-1}\tau^{m(r^2+s^2)}\mathsf{W}_{(r,s)}, \qquad 0 \leq m \leq d-1$$

Where we set (mod d): $\quad m = \dfrac{\beta}{2(1-\alpha)} \iff \alpha = \dfrac{4m^2-1}{4m^2+1}, \beta = \dfrac{4m}{4m^2+1} \quad (\Rightarrow \alpha^2 + \beta^2 = 1).$

These maps are well defined, as $4m^2 + 1 = 0$ (mod d) has no solution (-1 is no quadratic residue) for primes $d = 4k + 3$.

$\mathbf{F}$ corresponds, up to a phase, to $m = (d-1)/2$.

Balian *and* Itzykson *observed in 1986 that for primes* $d = 4k + 3$ *these matrices provide a unique common orthogonal basis of eigenvectors of* $\mathbf{F}$.

Theorem

Let $d = 4k + 3$ be prime. Let $\{ |\psi_i\rangle : 1 \leq i \leq d\}$ be the unique common orthogonal basis of eigenvectors of the d matrices

$$\mathbf{R}_m = \frac{1}{d} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \tau^{m(r^2+s^2)} \mathsf{W}_{(r,s)}, \quad 0 \leq m \leq d-1$$

The d^3 vectors $\{\mathsf{W}_{(k,l)} |\psi_i\rangle : 1 \leq i \leq d\, , 0 \leq k, l \leq d-1\}$ form a **projective 2-design**.

A strategy how to prove the Theorem

Let $\{\mathbf{P}_i := |\psi_i\rangle\langle\psi_i| : 1 \leq i \leq d\}$ be the projection matrices on the eigenvectors of the $\mathbf{R}_m$ and

$$\mathbf{P}_i^{(k,l)} = \mathsf{W}_{(k,l)}\mathbf{P}_i\mathsf{W}_{(k,l)}^{-1} \quad \text{with} \quad 0 \leq i, k, l \leq d-1$$

We are going to prove

$$\frac{1}{d^3}\sum_{i=0}^{d-1}\sum_{k=0}^{d-1}\sum_{l=1}^{d-1}\mathbf{P}_i^{(k,l)} \otimes \mathbf{P}_i^{(k,l)} = \frac{2}{d(d+1)}\mathbf{\Pi}_{sym}$$

For this we expand all $\mathbf{P}_i$ as well as $\mathbf{\Pi}_{sym}$ in terms of the Weyl-Heisenberg Matrices.

We sketch this on the next slides

Left side: $\mathbf{P}_i$ in terms of the Weyl-Heisenberg Matrices

Starting with the properties of $\mathbf{R}_m$ for $d = 4k + 3$:

$$tr\left(\mathbf{R}_m\right) = 1$$

$$tr\left(\mathbf{R}_m \mathbf{R}^*_{m'}\right) = \begin{cases} -1 & \text{if } m \neq m' \\ d & \text{if } m = m'. \end{cases}$$

Orthonormalization →

Auxiliary matrices $\mathbf{X}_m = \frac{1}{\sqrt{d+1}} \mathbf{R}_m + \kappa \mathbf{I}$, with $\kappa = \frac{\sqrt{d+1} - 1}{d\sqrt{d+1}}$

$$tr\left(\mathbf{X}_m\right) = 1$$

$$tr\left(\mathbf{X}_m \mathbf{X}^*_{m'}\right) = \begin{cases} 0 & \text{if } m \neq m' \\ 1 & \text{if } m = m'. \end{cases}$$

↓ Unitary map

One can derive the following form of $\mathbf{P}_i$ and conditions on the coefficients for $0 \leq r, s, i \leq d - 1$

$$\mathbf{P}_i = \frac{1}{d\sqrt{d+1}} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} p_i^{(r,s)} \mathbf{W}_{(r,s)} + \kappa \mathbf{I}$$

with $\quad p_i^{(0,0)} = 1, \qquad (p_i^{(r,s)})^* = p_i^{(-r,-s)}, \qquad \|(p_i^{(r,s)})_{0 \leq i \leq d-1}\|_2^2 = d$

Remark: When $\mathbf{P}_x$ is an *fiducial* projector for a Weyl-Heisenberg covariant SIC, then $|p_x^{(r,s)}|^2 = 1$.

Right side: SWAP **and $\mathbf{\Pi}_{sym}$ in terms of the Weyl-Heisenberg Matrices**

Siewert recently dedicated a paper to following statement and its consequences: Let $\{\mathbf{g}_{(r,s)}\}$, $0 \leq r, s \leq d-1$ be a basis of orthogonal matrices with norms $\|\mathbf{g}_{(r,s)}\|_2^2 = d$. Then $\mathsf{SWAP} = \frac{1}{d}\sum_{r=0}^{d-1}\sum_{s=0}^{d-1} \mathbf{g}_{(r,s)} \otimes \mathbf{g}^*_{(r,s)}$. As an application we notice

$$\mathsf{SWAP} = \frac{1}{d}\sum_{r=0}^{d-1}\sum_{s=0}^{d-1} \mathsf{W}_{(r,s)} \otimes \mathsf{W}_{(-r,-s)}$$

And as $\mathbf{\Pi}_{sym} = \frac{1}{2}(\mathsf{SWAP} + \mathbf{I} \otimes \mathbf{I})$

$$\mathbf{\Pi}_{sym} = \frac{1}{2}\left(\frac{1}{d}\sum_{r=0}^{d-1}\sum_{s=0}^{d-1} \mathsf{W}_{(r,s)} \otimes \mathsf{W}_{(-r,-s)} + \mathbf{I} \otimes \mathbf{I}\right)$$

Ostrovsky and Yakymenko gave an explicit proof of it in their paper (Geometric Properties of SIC-POVM Tensor Square).

„Magic“ of the Weyl-Heisenberg Matrices

$$\mathbf{P}_i = \frac{1}{d\sqrt{d+1}} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} p_i^{(r,s)} \mathsf{W}_{(r,s)} + \kappa \mathbf{I} \qquad \kappa = \frac{\sqrt{d+1}-1}{d\sqrt{d+1}}$$

$$p_i^{(0,0)} = 1, \qquad (p_i^{(r,s)})^* = p_i^{(-r,-s)}, \qquad \|(p_i^{(r,s)})_{0\leq i\leq d-1}\|_2^2 = d$$

$$\mathbf{\Pi}_{sym} = \frac{1}{2}\left(\frac{1}{d}\sum_{r=0}^{d-1}\sum_{s=0}^{d-1} \mathsf{W}_{(r,s)} \otimes \mathsf{W}_{(-r,-s)} + \mathbf{I}\otimes\mathbf{I}\right)$$

$$\frac{1}{d^3}\sum_{i=0}^{d-1}\sum_{k=0}^{d-1}\sum_{l=0}^{d-1} \mathsf{W}_{(k,l)}\mathbf{P}_i\mathsf{W}_{(k,l)}^{-1} \otimes \mathsf{W}_{(k,l)}\mathbf{P}_i\mathsf{W}_{(k,l)}^{-1} = \frac{2}{d(d+1)}\mathbf{\Pi}_{sym}$$

The rest is just application of relations on Weyl-Heisenberg Matrices, especially

$$\mathsf{W}_{(k,l)}\mathsf{W}_{(r,s)} = \tau^{2(rl-sk)}\mathsf{W}_{(r,s)}\mathsf{W}_{(k,l)} = \tau^{(rl-sk)}\mathsf{W}_{(k+r,l+s)}$$

Remark on 2-designs

Slomczynski and Szymusiak noticed, that $\{\mathbf{P}_i := |\psi_i\rangle\langle\psi_i| : 1 \leq i \leq N\}$ is a 2-design, iff for any matrix $\boldsymbol{\rho}$ with $\mathrm{tr}(\boldsymbol{\rho}) = 1$

$$\boldsymbol{\rho} = (d+1)\sum_{i=1}^{N} \frac{d}{N}\mathrm{tr}(\boldsymbol{\rho}\mathbf{P}_i)\mathbf{P}_i - \mathbf{I}$$

A short sketch of the proof:

We use the equation

$$\sum_{i=1}^{N} \mathbf{P}_i \otimes \mathbf{P}_i = \frac{2N}{d(d+1)}\boldsymbol{\Pi}_{sym} = \frac{N}{d(d+1)}\left(\mathbf{I}\otimes\mathbf{I} + \mathsf{SWAP}\right).$$

Multiplying it with $\mathbf{I}\otimes\boldsymbol{\rho}$, applying the partial trace, and using $\mathrm{tr}_{[2]}\left((\mathbf{I}\otimes\boldsymbol{\rho}).\mathsf{SWAP}\right) = \boldsymbol{\rho}$ (see Siewert), we get

$$\sum_{i=1} \mathrm{tr}(\boldsymbol{\rho}\mathbf{P}_i)\mathbf{P}_i = \frac{N}{d(d+1)}(\mathrm{tr}(\boldsymbol{\rho})\mathbf{I} + \boldsymbol{\rho})$$

As a consequence follows

Another *„Measurement on the Sky“*

Let $d = 4k + 3$ be prime. Let $\{\mathbf{P}_i := |\psi_i\rangle\langle\psi_i| : 1 \leq i \leq d\}$ be the projection matrices on the eigenvectors of the $\mathbf{R}_m$, and $\mathbf{P}_i^{(k,l)} = \mathsf{W}_{(k,l)}\mathbf{P}_i\mathsf{W}_{(k,l)}^{-1}$ with $0 \leq i, k, l \leq d - 1$.

Then for any matrix $\boldsymbol{\rho}$ with $\mathrm{tr}(\boldsymbol{\rho}) = 1$

$$\boldsymbol{\rho} = (d+1)\sum_{i=0}^{d-1}\sum_{j=0}^{d-1}\sum_{k=0}^{d-1}\rho_i^{(k,l)}\mathbf{P}_i^{(k,l)} - \mathbf{I} \qquad \text{with} \qquad \rho_i^{(k,l)} = \frac{1}{d^2}\mathrm{tr}(\boldsymbol{\rho}\mathbf{P}_i^{(k,l)})$$

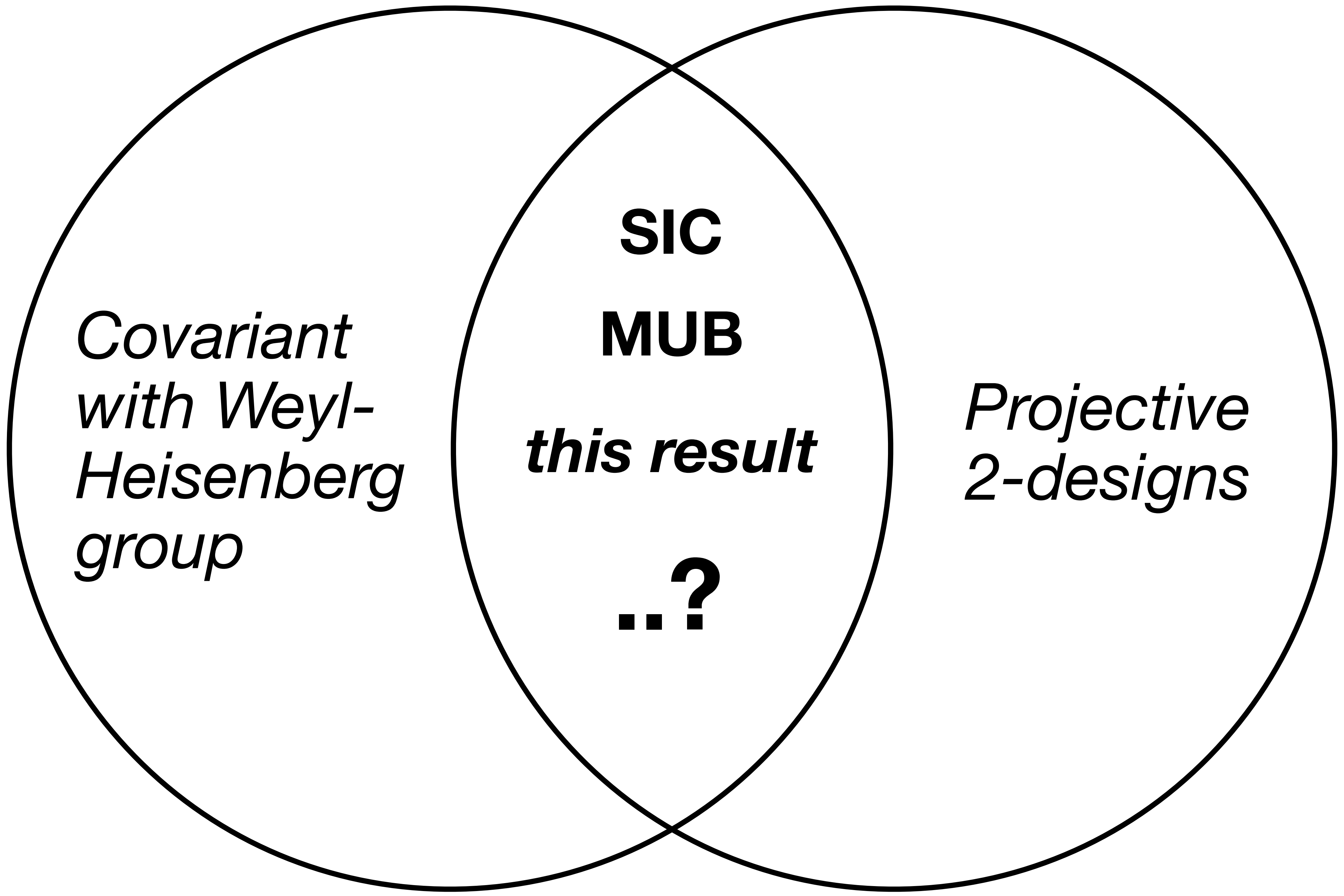
This result can be seen as counterpart of the equation for **SIC**

$$\boldsymbol{\rho} = (d+1)\sum_{i=1}^{d^2}\rho_i\mathbf{P}_i - \mathbf{I} \qquad \text{with} \qquad \rho_i = \frac{1}{d}\mathrm{tr}(\boldsymbol{\rho}\mathbf{P}_i)$$

which attracted a great deal of attention in the context of the **QBism** approach to quantum theory by `Fuchs`, et al.

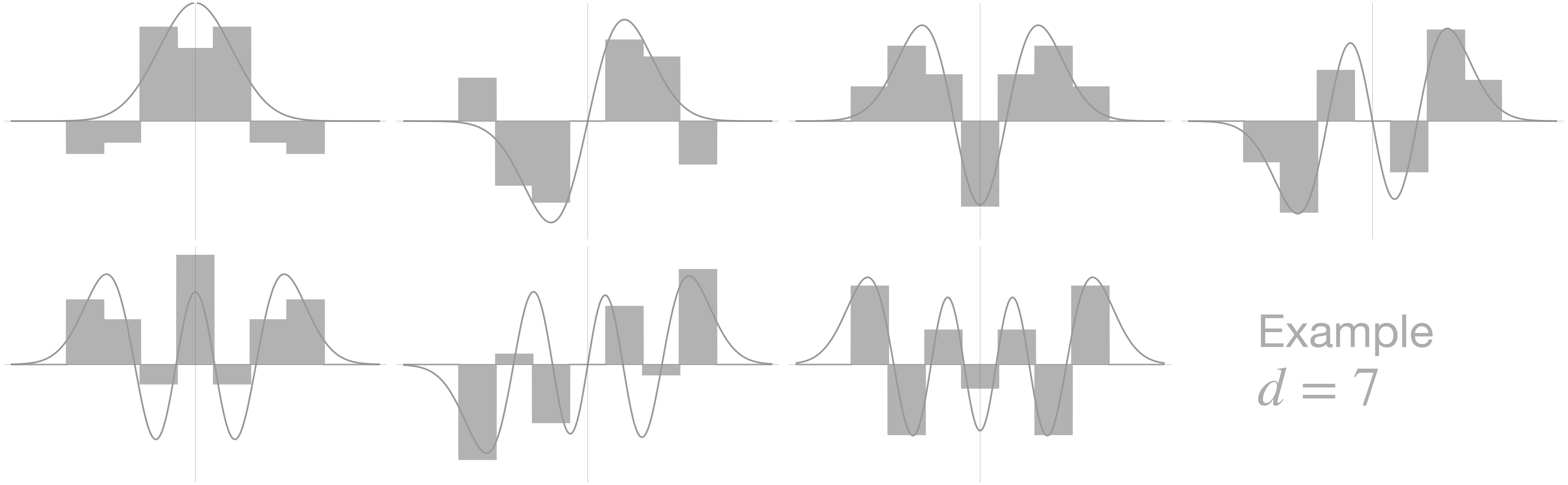
Possible further research

- Similar 2-designs of (subsets) of eigenvectors of $\mathbf{F}$ in other dimensions d ?
- Starting with other matrices ?



Sets of rank 1 projection matrices

Discrete (finite) vs. Continuum …



As d grows these eigenvectors of the Fourier matrix have no clear/strong convergence to the harmonic oscillator eigenstates. Does 2-design property offer another approach?

Thank you
Dziękuję!